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Finite-temperature magnetism in fractional quantum Hall systems: the composite-fermion Hartree–Fock approximation and beyond

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Abstract. Using the Hamiltonian formulation for composite fermions developed recently, the temperature dependence of the spin polarization is computed for the translationally invariant fractional quantum Hall states at $\nu = 1/3$ and $\nu = 2/5$ in two steps. In the first step, the effect of particle–hole excitations on the spin polarization is computed in a composite-fermion Hartree–Fock approximation. The computed magnetization for $\nu = 1/3$ lies above the experimental results for intermediate temperatures indicating the importance of long-wavelength spin fluctuations which are not correctly treated in the Hartree–Fock approximation. In the second step, spin fluctuations beyond the Hartree–Fock approximation are included for $\nu = 1/3$ by mapping the problem onto the coarse-grained continuum quantum ferromagnet. The parameters of the description in terms of the effective continuum quantum ferromagnet are extracted from the preceding Hartree–Fock analysis. After the inclusion of spin fluctuations in a large-N approach, the results for the finite-temperature spin polarization are in quite good agreement with the experiments.

1. Introduction

The fractional quantum Hall (FQH) effect [1] has introduced us to new, highly correlated, incompressible states [2] of electrons in high magnetic fields. A unified understanding of all fractions $\nu = p/(2sp+1)$ was achieved by the composite-fermion picture of Jain [3], in which the electrons are dressed by $2s$ units of statistical flux to form composite fermions (CFs). At a mean-field level, the CFs 'see' a reduced field $B^* = B/(2sp + 1)$, in which they fill p CF Landau levels (CF-LLs), and exhibit the integer quantum Hall effect.

Due to the small g-factor of electrons in GaAs, spins may not be fully polarized in FQH states [4, 5]. Transitions between singlet, partially polarized, and fully polarized states (based on gap measurements) have been observed for a number of fillings [6–9], which can be understood in terms of CFs with a spin [3, 9, 10]. The transitions happen when an unoccupied CF-LL of one spin crosses the occupied CF-LL of the opposite spin.

While these low-temperature measurements are in satisfactory agreement with the ground states predicted in the CF picture [10], in order to understand the temperature dependence of the polarization $P(T)$ one has to consider all excited states as well. Detailed measurements of $P(T)$ for the $\nu = 1/3$ state have recently appeared in the literature [11, 12]. It is well known that the $\nu = 1/3$ state is spontaneously polarized at $T = 0$, even when the Zeeman coupling $E_Z = g\mu B_{tot}$ is zero. In this it is analogous to the $\nu = 1$ state [13], which has been extensively studied theoretically [14–16] and experimentally [17]. There are, however, significant differences between the two cases at finite T.

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In a recent paper, MacDonald and Palacios [18] identified a key qualitative feature that makes $\nu = 1/3$ very different from $\nu = 1$. In the $\nu = 1$ case the particle–hole excitations are very high in energy compared to E_Z, and are frozen out at all low temperatures of interest. Consequently, the T-dependence of P comes mainly from spin-wave excitations and their interactions. This is why long-wavelength effective theories such as the continuum quantum ferromagnet [14] approach are successful. However, for $\nu = 1/3$, particle–hole excitations are on the same scale of energy as E_Z, and cannot be ignored at any T. MacDonald and Palacios use a simplified model to illustrate this feature [18], but the model is not sufficiently detailed to enable a calculation of $P(T)$ for a realistic sample, and is difficult to extend to non-Laughlin fractions.

The goal of this paper is to describe a general analytical method for approximately computing $P(T)$ for an arbitrary principal fraction for realistic samples. To this end, an approximate Hamiltonian formalism in which composite-fermion variables explicitly appear will be used [19]. This Hamiltonian approach is based on the field theoretic idea of attaching flux to electrons by using a Chern–Simons field [20–26]. The CF Hamiltonian approach has many features suited to the computation of the physical properties of fractional Hall systems. The CFs 'see' the effective field, and fill CF Landau levels (CF-LLs). For the principal fractions $\nu = p/(2p+1)$ with which we will be concerned, the effective field is $B^* = B/(2p+1)$, and the CFs fill p CF-LLs. The energies of the CF-LLs are controlled entirely by interactions [19], which is a correct feature of the physics of electrons in the lowest Landau level. Note that this is *not* a theory in which composite fermions are free. On the contrary, the theory is fully interacting, with both the kinetic energies and the residual interactions of the CFs being controlled by the electron–electron interaction. Finally, the nonperturbative charge and dipole moment of the excitations appear explicitly in the theory [19]. The fact that all of these nontrivial features are built into the theory raises the expectation that very simple approximate treatments of this Hamiltonian theory (such as the Hartree–Fock one) will suffice to produce reasonable numbers for physical quantities. This expectation is indeed borne out by explicit calculations [27–29]. The Hamiltonian formalism will be presented briefly in section 2.

The Hamiltonian approach is particularly suited to the computation of finite-temperature properties in fractional Hall systems. In the composite-fermion Hartree–Fock (CFHF) approximation, one self-consistently finds the single-particle energies and occupations of the various CF-LLs at any finite temperature. Since the energies of the states are controlled entirely by interactions and occupations, *these energies will be temperature dependent*. This is a familiar feature in other interacting many-body systems, such as the BCS superconductor, where the single-particle gap is a collective effect, and depends on temperature. Thus the Hamiltonian theory gives us valuable information on the evolution of the collective state as a function of temperature, which is then used in the mapping to the effective theory. Once the occupations of the single-particle CF-LLs have been determined, the polarization is computed simply as the difference of the total $\uparrow$-spin and $\downarrow$-spin occupations. This yields the CFHF prediction for the temperature dependence of the spin magnetization. This procedure and the results for $\nu = 1/3$ and $\nu = 2/5$ are described in section 3. A brief description of this work has appeared previously [30].

It turns out that for the spontaneously polarized $\nu = 1/3$ state the CFHF prediction is higher than the experimental values for all temperatures, with the discrepancy being considerable for intermediate temperatures, as shown in figure 1. To put it in the proper context, the agreement between the CFHF prediction and experiment is considerably better than for $\nu = 1$, where the HF approximation does a very poor job of predicting the spin polarization [15].

However, the discrepancy is nonetheless there, and is presumably the result of spin fluctuations which are not treated correctly in the CFHF approximation. One can imagine

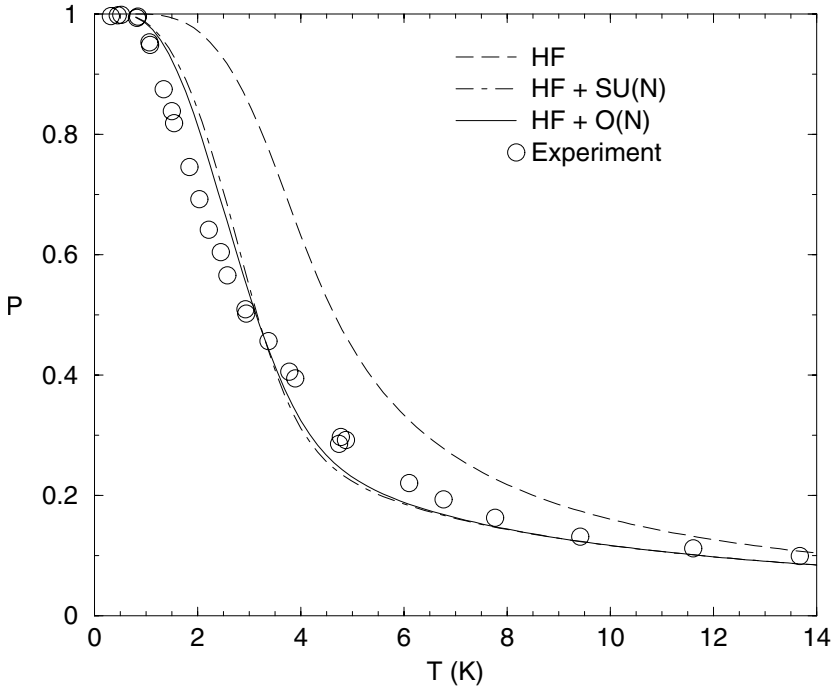

Figure 1. Polarization versus T for $\nu = 1/3$. The circles are the data for the 10W sample of Khandelwal *et al* [11]. The dashed line is the prediction from the CFHF theory for a thickness parameter of $\lambda = 1.5l$. The solid and dot–dashed lines refer to the theoretical prediction including spin fluctuations.

integrating out the fermions to obtain an effective theory that has low-energy spin degrees of freedom. One is then led to map the low-energy physics onto the continuum quantum ferromagnet (CQFM) [14]. The CQFM has two free parameters, the magnetization per unit volume M_0, and the spin stiffness ρ_s. In the traditional CQFM theory these are temperature-independent parameters. However, since the theory is fermionic at the microscopic level, with the fermionic energy levels and occupations being temperature dependent, one should expect M_0 and ρ_s to acquire a T-dependence in the effective theory. It turns out that these parameters can be easily extracted from the CFHF treatment of section 3. Armed with this information, we proceed to include spin fluctuations in a large-N approach as described by Read and Sachdev [14]. The results are in quite good agreement with the experimental data over the whole range of temperature, as shown in figure 1 for the 10W sample of Khandelwal *et al* [11]. The same comparison is shown for the data of Melinte *et al* [12] in figure 2.

The details of the figures will be explained in sections 3 and 4. The mapping to the effective theory and the subsequent calculations are described in section 4. We end with some conclusions, caveats, and open questions.

2. Hamiltonian formalism

Since detailed descriptions of the Hamiltonian theory of CFs have appeared elsewhere [19], we will restrict ourselves to a summary of the essential features of this formalism.

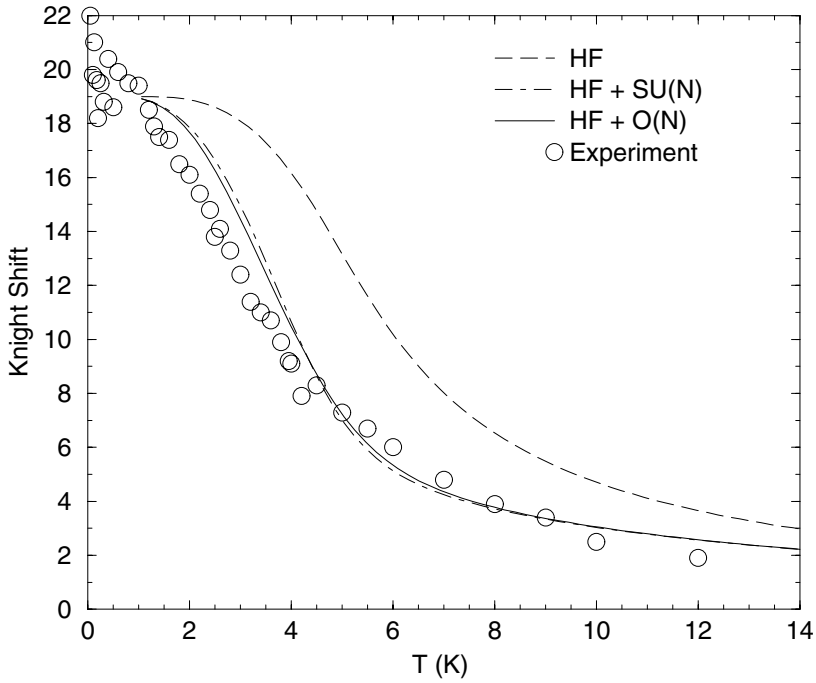

Figure 2. The theoretical prediction in the CFHF approximation (dashed line) and including spin fluctuations (solid and dot–dashed lines) compared to the Knight shift data for the M242 sample of Melinte *et al* [12]. A saturation value of 19 kHz for the Knight shift has been used, and the thickness parameter is assumed to be $\lambda = 1.5l$.

The composite-fermion picture was originally used by Jain [3] to generate electronic wave functions with good correlations. These wave functions have excellent overlap with the exact wave functions for finite systems, and encode all the right physics. In order to compute dynamical response functions, it is desirable to have an operator or field theoretic formulation of CFs. The fundamental property that CFs carry statistical flux can be implemented by a Chern–Simons (CS) transformation, which performs flux attachment via the CS gauge field to obtain a field theoretic description with either bosons [20–23] or fermions [24]. These theories have provided us with a link between the microscopic formulation of the problem and experiment, both for incompressible and compressible states [25, 26].

Recently R Shankar and the present author developed a Hamiltonian CS theory for the fractional quantum Hall states [19]. Inspired by the work of Bohm and Pines [31] on the 3D electron gas, we enlarged the Hilbert space to introduce n high-energy magnetoplasmons degrees of freedom (n also being the number of electrons), at the same time imposing an equal number of constraints on physical states. However, the fermions still had the bare mass (recall that in the lowest Landau level, the electrons should lose all memory of the bare mass, and acquire an effective mass controlled by interactions), and the frequency of the magnetoplasmons was incorrect. Hence a final canonical transformation was employed to decouple the fermions from the oscillators *in the infrared limit*.

The final fermions are called *the* composite fermions for the following reasons. Firstly, the final fermions have no dispersion in the absence of interactions and acquire an effective mass dependent on interactions alone. Next, the final canonical transformation assigns to each fermion the magnetic moment $e/2m$ as mandated by very general arguments [32, 33]. The

central result of the formalism is the formula for the electronic charge density, which takes the following form, separable into high- and low-energy pieces [19], at small q:

$$\rho_e(q) = \frac{q}{\sqrt{8\pi}}\sqrt{\frac{2p}{2p+1}}(A(q)+A^\dagger(-q)) + \left(\sum_j \mathrm{e}^{-\mathrm{i}qx_j}\right)\Big/(2p+1) - \mathrm{i}l^2\left(\sum_j (q\times\Pi_j)\mathrm{e}^{-\mathrm{i}qx_j}\right) \tag{1}$$

where A, $A^\dagger$ refer to the annihilation and creation operators of the magnetoplasmon oscillators, $l = 1/\sqrt{eB}$ is the magnetic length, and $\vec{\Pi}_j = \vec{P}_j + e\vec{A}^*(r_j)$ is the velocity operator of the CFs. The oscillator piece saturates Kohn's theorem [35]. The rest, to be called $\bar{\rho}$, is obtained by adding to the canonically transformed electronic charge density a particular multiple of the constraint [19] (in the physical subspace, one can add any multiple of the constraint without physical consequences, but we wish to work in the full space). It has some very useful properties in the full space:

- $\bar{\rho}$ satisfies the magnetic translation algebra (MTA) [36] to lowest leading order. Since this is the algebra of the electron density in the lowest Landau level (LLL), the LLL projection has been correctly carried out in the infrared.
- Note that $\bar{\rho}$ is a sum of a monopole with charge $e^* = e/(2p+1)$, which is the charge associated with the CF, and a dipole piece which alone survives at $\nu = 1/2$ and has the value proposed by Read [34]. (A number of recent constructions have emphasized this dipolar aspect [37–40].)
- We also find that as $\vec{q} \to 0$ all transition matrix elements of $\bar{\rho}$ from the HF ground state vanish at least as q^2.

The final property is an essential property of *physical* charge-density matrix elements from incompressible liquid ground states in the LLL [36]. It is easy to see that if one intends to use the Hartree–Fock approximation ignoring constraints, these properties of $\bar{\rho}$ are essential. They make it plausible that $\bar{\rho}$ does not suffer vertex corrections.

The Hamiltonian of the low-energy sector (dropping the magnetic moment term) is

$$H = \frac{1}{2}\int \frac{\mathrm{d}^2q}{(2\pi)^2}\, v(q)\bar{\rho}(-q)\bar{\rho}(q). \tag{2}$$

where $v(q)$ is the electron–electron interaction. Real samples have a finite thickness Λ of the same order as l, so the Coulomb interaction is cut-off at large wave vectors [41]. We will model this interaction by [42]

$$v(q) = \frac{2\pi e^2}{q}\exp -\lambda q. \tag{3}$$

Finally, the constraint will be approximately implemented by cutting off the number of CF-LLs to maintain the correct number of *electronic* states. For $\nu = p/(2p+1)$ this means keeping $2p+1$ CF-LLs. More details of the formalism can be found in references [19, 29].

3. The composite-fermion Hartree–Fock approximation

The CFHF approximation has been applied to the above Hamiltonian, and reasonable success has been obtained in computing various physical quantities in the gapped fractions [27–29], including a very recent calculation of the temperature-dependent polarization $P(T)$ for the compressible half-filled LL [43].

Before one employs the CFHF approximation, one needs to express the Hamiltonian as an operator acting on a set of states. Since the CFs 'see' the effective field B^*, it is natural to represent the Hilbert space as Slater determinants of single-particle CF Landau level states in the effective field. The wave functions of these states in the Landau gauge are

$$\phi_{n,X}(\vec{r}) = \frac{1}{\sqrt{L}} \mathrm{e}^{\mathrm{i}Xy/(l^*)^2} \phi_n((x - X)/l^*) \tag{4}$$

Here $l^* = l\sqrt{2p+1}$ is the effective magnetic length, L is the linear dimension of the system, $X = 2\pi(l^*)^2 j/L$ is the degeneracy index, where $j = 0, 1, \ldots, L^2/(2\pi(l^*)^2)$, and ϕ_n is the nth normalized harmonic oscillator eigenfunction. The index n is the CF-LL index.

The density is a one-body operator, and can be expressed in this basis (after the spin labels have been included) as

$$\hat{\tilde{\rho}}(\vec{q}) = \sum_{\sigma, X\{n_i\}} \mathrm{e}^{-\mathrm{i}q_x X} \rho_{n_1 n_2}(\vec{q}) d^\dagger_{\sigma, n_1, X - q_y (l^*)^2/2} d_{\sigma, n_2, X + q_y (l^*)^2/2} \tag{5}$$

where $\sigma = \uparrow, \downarrow$ is the spin index, and $d_{\sigma,n,X}$ destroys a CF in the single-particle state labelled by σ, n, X, and $\rho_{n_1 n_2}(\vec{q})$ is a matrix element given by

$$\rho_{n_1 n_2}(\vec{q}) = \frac{(-1)^{n_< + n_2}}{2p+1} \sqrt{\frac{n_<!}{n_>!}} \mathrm{e}^{\mathrm{i}(\theta_q - \pi/2)(n_1 - n_2)} \left(\frac{ql^*}{\sqrt{2}}\right)^{n_> - n_<} \times \mathrm{e}^{-y/2} (n_> L^{n_> - n_<}_{n_< - 1} + 2L^{n_> - n_<}_{n_<} - (n_< + 1) L^{n_> - n_<}_{n_< + 1}) \tag{6}$$

where $n_<$ ($n_>$) is the lesser (greater) of n_1, n_2, θ_q is the angle of the vector $\vec{q}$ measured from the x-axis, $y = (ql^*)^2/2$ is the argument of the Laguerre polynomials L^k_n.

The Hamiltonian is now a four-fermion operator. It can easily be shown that the single-particle states defined above form a good HF basis [29]. In the CFHF approximation, one reduces the four-fermion Hamiltonian to a two-fermion operator by taking averages according to the rules

$$\langle \mathrm{GS}| d^\dagger_\nu d_{\nu'} |\mathrm{GS}\rangle = \delta_{\nu\nu'} N_F(\nu) \tag{7}$$

$$\langle \mathrm{GS}| d_\nu d^\dagger_{\nu'} |\mathrm{GS}\rangle = \delta_{\nu\nu'} (1 - N_F(\nu)) \tag{8}$$

where ν is a shorthand for all the state labels. This results in the HF Hamiltonian

$$H_{HF} = \sum_{\sigma,n,X} \epsilon(\sigma, n) d^\dagger_{\sigma,n,X} d_{\sigma,n,X} \tag{9}$$

where the HF single-particle energy is

$$\epsilon(\sigma, n) = \pm \frac{E_Z}{2} + \frac{1}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, v(q) \sum_m (1 - 2N_F(\sigma, m)) |\rho_{nm}(q)|^2 \tag{10}$$

in which the Zeeman energy has been added ($E_Z = g^* \mu_B B$).

Finding the energies at $T = 0$ is quite simple, since the occupations of the CF-LLs can only be 0 or 1. For example, for the $\nu = 1/3$ state, $N_F(\uparrow, 0) = 1$ and all other occupations are zero, and for the $\nu = 2/5$ singlet state, $N_F(\uparrow, 0) = N_F(\downarrow, 0) = 1$, and all other occupations are zero. For nonzero T one has to carry out a self-consistent procedure. First one chooses trial values for the energies (say the $T = 0$ values) and a trial value for the chemical potential μ (say halfway between the lowest unoccupied and highest occupied CF-LL). Then one assigns the occupations of the single-particle levels according to Fermi occupation function

$$N_F(\sigma, n) = \frac{1}{1 + \exp(\epsilon(\sigma, n) - \mu)/T} \tag{11}$$

with the trial value of μ. Then one recomputes the HF energies using equation (10). Note that the structure of degenerate CF-LLs remains intact, and only the energies and occupations change. Since the filling has to remain fixed, the chemical potential will change when the energies change. One then recomputes the chemical potential as the root of the equation

$$\sum_n (N_F(\uparrow, n) + N_F(\downarrow, n)) = p \tag{12}$$

for the principal fraction $p/(2p+1)$ and iterates the whole process until self-consistency is achieved. Finally, the spin polarization is given by

$$P = \sum_n (N_F(\uparrow, n) - N_F(\downarrow, n))/p. \tag{13}$$

From the above procedure it is clear that only single-particle excitations have been taken into account in obtaining $P(T)$ so far, and spin fluctuations have been explicitly ignored. If it so happens that for the system under consideration the effects of spin fluctuations are small, then this approximation should be accurate, otherwise not.

Let us proceed to compare the CFHF results to experiments. We first consider the 10W sample of Khandelwal *et al* [11]. The sample parameters are $B_\perp = 9.61$ T, and $B_{tot} = 12$ T. This implies that the Coulomb energy scale is $E_C = e^2/\varepsilon l \approx 160$ K and the Zeeman energy is $E_Z = 0.0175 E_C$. We will use a value of $\lambda = 1.5l$ for the thickness parameter for illustrative purposes throughout this paper. This value ought to be in the physical regime [44] for most samples, and also approximately agrees with that extracted from an analysis of the compressible states [43]. *It should be emphasized that the CFHF analysis, and the mapping to the effective spin theory that follows, can be performed for any potential* $v(q)$.

Figure 1 (in the introduction) shows the HF prediction from our theory for $\lambda = 1.5l$ compared to the experimental data. The agreement is good at very low ($T \leqslant 1$ K) and very high ($T \geqslant 8$ K) temperatures. However, at intermediate T, there is a big discrepancy between the CFHF prediction and the data. This indicates that effects not treated correctly in the HF approximation, notably long-wavelength thermal spin fluctuations, are important in this intermediate regime of T. Nonetheless, to put the result in context, one should note that the CFHF prediction agrees much better with the data than the corresponding HF prediction for $\nu = 1$ (see, for example, reference [15] for a comparison of the different predictions for $\nu = 1$). This is because at $\nu = 1$ particle–hole excitations are completely unimportant at all temperatures of interest, while thermal spin fluctuations dominate. Since long-wavelength spin fluctuations are treated very poorly in the HF approximation the agreement is bad. However, at $\nu = 1/3$, particle–hole excitations play a major role in reducing $P(T)$ for $T > 5$ K. In the next section we will see how to incorporate spin fluctuations into our calculation, resulting in much better agreement with the data.

Figure 3 shows the same type of comparison for the Knight shift data of Melinte *et al* [12] for their M242 sample. Here the sample parameters are $B_{tot} = B_\perp = 17$ T. This implies that $E_C \approx 210$ K, $E_Z = 0.0186 E_C \approx 4$ K. Once again a value of $\lambda = 1.5l$ has been used in the CFHF calculation. In order to translate the Knight shifts into polarization numbers or vice versa, one needs to determine the saturation value of the Knight shift, which presumably corresponds to a polarization of $P = 1$. There is a lot of scatter in the data at low T, due to the very long times needed to measure the Knight shift, and the error bars are also large at low T [12]. There is thus some uncertainty in determining the Knight shift corresponding to $P = 1$. In any reasonable theory one expects to find that $P = 1$ for $T \ll E_Z$, and expects to see this saturated value of P up to about $T = 0.5 E_Z$ or so.

On the basis of these considerations two values of the saturation Knight shift $K_{s,P=1}$, 21 kHz and 19 kHz, have been used to fit the data here, both of which lie within the error bars

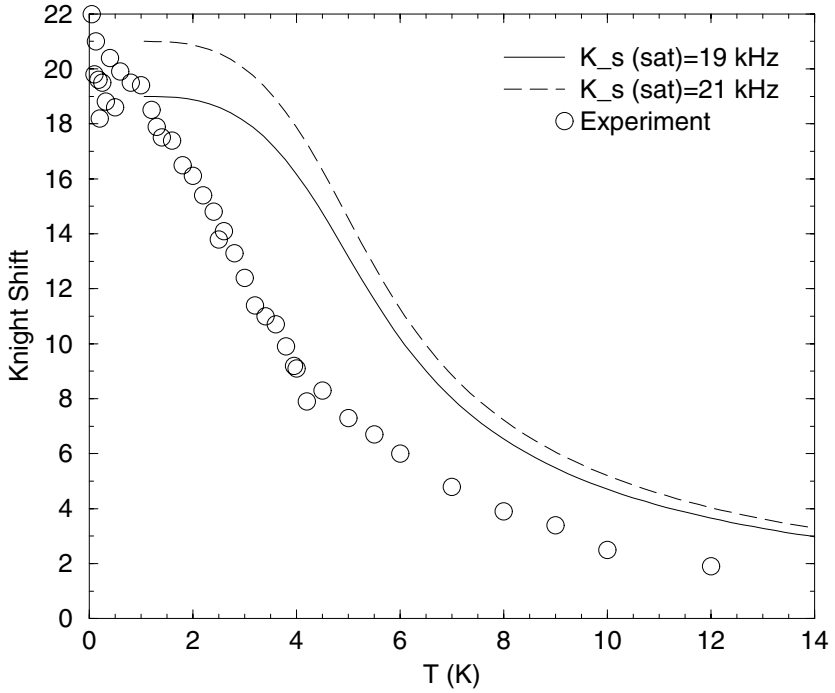

Figure 3. Knight shift versus T for $\nu = 1/3$. The circles are the data for the M242 sample of Melinte *et al* [12]. The lines are the predictions from the CFHF theory for $\lambda = 1.5l$, assuming two different values for the Knight shift that corresponds to saturated polarization.

of the low-T data [12]. One possibility that can explain this spread is that spin-reversed quasiparticles are present in the ground state due to disorder, which can bring down the 'saturated' value of the Knight shift [45]. The 21 kHz value was used by Melinte *et al* in a phenomenological $\tanh(\Delta/4k_BT)$ fit to obtain $\Delta = 1.7E_Z$. However, one must note that the fit for $K_{s,P=1} = 19$ kHz seems slightly better, since then the experimental saturation region is about $0.5E_Z$. The agreement between theory and experiment for this value of $K_{s,P=1}$ is also better than for $K_{s,P=1} = 21$ kHz. The 19 kHz value will be used in the mapping to the effective theory in the next section. Overall the agreement is slightly worse than for the Khandelwal *et al* data [11], but leads to the same conclusion: it is quite important to treat thermal spin fluctuations correctly at intermediate temperatures for $\nu = 1/3$.

Before we proceed to approximately incorporate spin fluctuations into the theoretical prediction, let us address the following interesting question: why is the HF approximation so good in this case relative to the case of $\nu = 1$? To answer this question let us turn to the spin-wave dispersions. The spin wave is a collective spin-flip excitation, and at wave vector $\vec{q}$ corresponds a plane-wave state in which a majority-spin quasihole and a minority-spin quasiparticle are at a separation of $q(l^*)^2$.

The $q \to 0$ limit is required to be E_Z by Larmor's theorem, while in the $q \to \infty$ limit the particle and hole become infinitely separated, so the energy of the excitation is the spin-reversed particle–hole gap Δ_{SR}. The dispersion of these excitations can be computed for $\nu = 1/3$ in the manner described in reference [29], and is shown in figure 4 for $\lambda = 1.5l$ for $E_Z = 0.0175E_C$ and $T = 0$. Figure 4 explicitly illustrates the feature [18] that the spin-flip particle–hole excitations are at the same energy scale as E_Z.

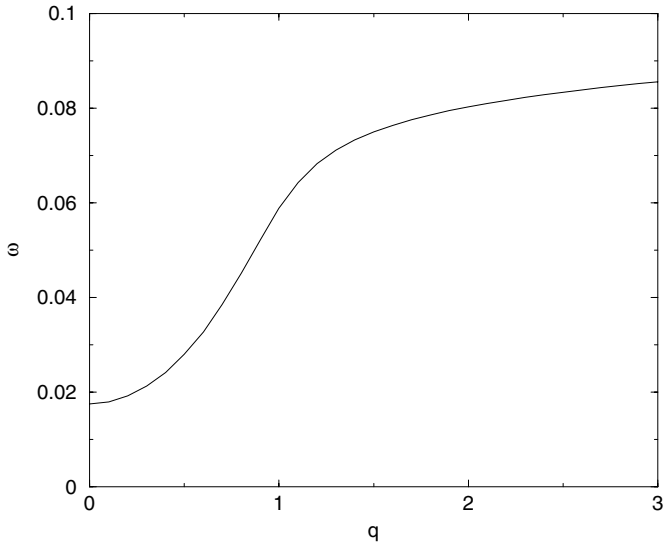

Figure 4. Spin-wave energy dispersions in units of the Coulomb energy E_C for $E_Z = 0.0175$ and $\lambda = 1.5l$, with q plotted in units of l^{-1}. As can be seen, the scale of the spin-reversed gap is the same as E_Z.

How does this evolve as temperature increases? Recall that all the energy splittings in the CF Hamiltonian formalism come from interactions, and as the occupations of the states change with T so do their HF energies. As T increases, the occupations of the minority-spin levels increase while that of the lowest majority-spin level decreases. This means that the exchange splitting between the minority- and majority-spin levels *decreases*, as can be seen from equation (10). It is clear that as T becomes very large the occupations of all the levels should tend to become the same, and therefore Δ_{SR} should tend towards E_Z. Since the spin-wave dispersion has to be E_Z for very small q, and Δ_{SR} for very large q, this implies that the spin-wave dispersion becomes increasingly flat at T increases. We will estimate the spin stiffness in the next section and corroborate this conclusion. Figure 5 shows this behaviour of Δ_{SR} explicitly for the same parameters as in figure 4.

Now consider a *noninteracting* theory of CFs. This would have a completely flat dispersion for the collective mode, that is, $\omega(q) = E_Z$. Therefore, as the temperature increases, the theory appears to be more weakly interacting, and the CFHF approximation becomes more accurate. This trend can be expected to continue until a temperature scale is reached where CFs cease to exist. There are no obvious signs of such a scale in the data.

One qualitatively incorrect feature of the HF approximation in general [15], which is shared by the CFHF approximation used here, is that it predicts a magnetically ordered state for nonzero T, and a finite-temperature phase transition, even in the absence of any Zeeman coupling. The CFHF predictions for $E_Z = 0$ and three values of λ are shown in figure 6. This failure of the CFHF approximation highlights the need for a proper treatment of spin fluctuations.

The CF Hamiltonian theory and the CFHF approximation are very general, and can be applied to any fractional Hall state (for an application to compressible states see reference [42]). To illustrate this figure 7 shows the $P(T)$ curves for $\nu = 2/5$ for $\lambda = 1.5l$ for a range of Zeeman couplings. Note the nonmonotonicity of the curves that start from the singlet ground state at $T = 0$. There is a transition to the fully polarized state around $E_Z = 0.01E_C$.

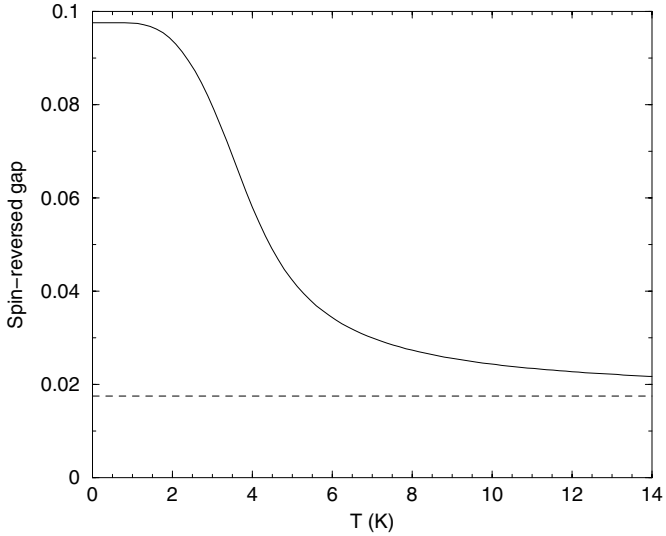

Figure 5. Spin-reversed gap Δ_{SR} in units of E_C as a function of T. The dashed line is E_Z. Beyond about 6 K, Δ_{SR} is essentially the same as E_Z.

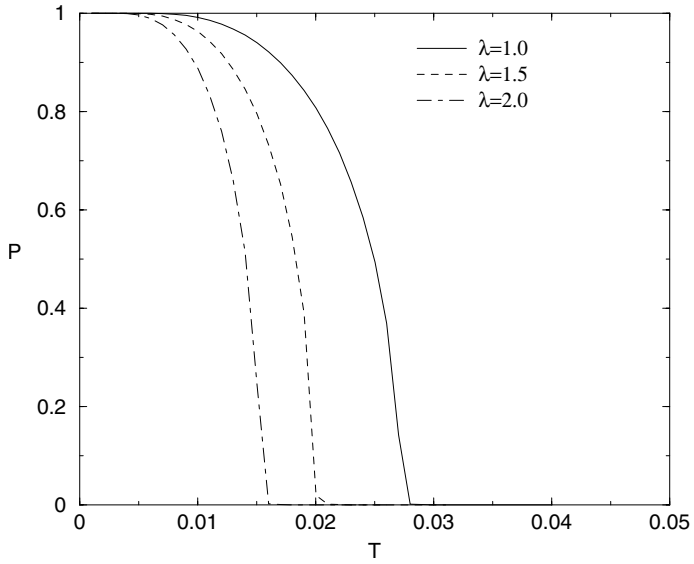

Figure 6. Polarization versus T for $\nu = 1/3$ for three different values of λ for $E_Z = 0$. T is plotted in units of the Coulomb interaction E_C.

Note also that only results for translationally invariant HF states are presented, which ignores possible partially polarized *inhomogeneous* states that have recently been proposed [46] to explain intriguing observations by Kukushkin *et al* [47] of a state with half the maximal polarization for $\nu = 2/5$, which is not allowed as a translationally invariant CF state at $T = 0$.

Let us compare our results to the only other method that can compute $P(T)$ for arbitrary fractions, which is exact diagonalization (keeping all the excited states) and subsequent calculation of thermodynamic quantities [48]. Due to computational limitations, this method is

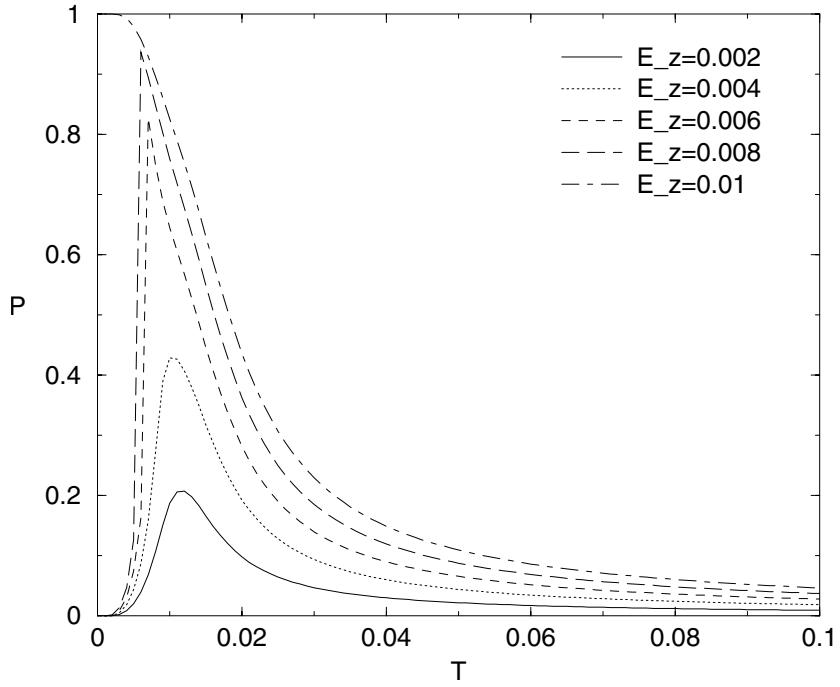

Figure 7. Polarization versus T for $\nu = 2/5$ and $\lambda = 1.5l$, for a range of values of E_Z. E_Z and T are both plotted in units of E_C.

restricted to fairly small systems. For example, the largest system studied by Chakraborty and Pietilainen [48] for $\nu = 1/3$ has five electrons, and for $\nu = 2/5$ has four electrons. For $\nu = 1/3$ the exact-diagonalization result lies above our CFHF predictions (and the experiment) for $T > 8$ K. This discrepancy might be the result of finite-thickness and/or finite-size corrections. However, at low T the exact-diagonalization result [48] follows the data more closely than our HF approximation (in all of the above comparisons the $g = 0.5$ line in figure 1 of reference [48] has been used and compared to the 10W sample of Khandelwal *et al* [11]; this sample has the closest parameters to those used in reference [48]). For $\nu = 2/5$, the CFHF results reproduce the nonmonotonicity of $P(T)$ for those values of E_Z where the singlet state is the ground state, and the peaks in $P(T)$ occur at roughly the same T in the CFHF results and the exact-diagonalization results [48]. However, the same overall pattern holds for $\nu = 2/5$, namely, the results of Chakraborty and Pietilainen [48] are below the CFHF prediction for low T, but are higher for $T > 0.02E_C$, where they once again see a $1/T$ tail with a large coefficient. It would be interesting to explore the finite-size systematics to see if the large-T tail is suppressed for larger sizes.

4. Spin fluctuations: mapping to the continuum quantum ferromagnet

As can be seen in figures 1 and 2, spin fluctuations are quite important for $\nu = 1/3$ at intermediate temperatures. Also, as seen in figure 6, the HF approximation has an incorrect qualitative feature, in that it predicts a magnetized state at nonzero T even in the absence of a Zeeman coupling, which also points to the need to include spin fluctuations. For $\nu = 1$, the coarse-grained effective theory of the continuum quantum ferromagnet (CQFM) coupled with

the large-N approximation, first applied to this problem by Read and Sachdev [14], has been quite successful in explaining the temperature dependence of the polarization. In this section a method is presented to map the $\nu = 1/3$ problem to the CQFM.

In the CQFM description one starts with the action [14]

$$S = \int \mathrm{d}^d x \int_0^{1/T} \mathrm{d}\tau \left(\mathrm{i} M_0 \vec{A}(\vec{n}) \cdot \nabla_\tau \vec{n} + \frac{\rho_s}{2} (\nabla_x \vec{n})^2 - M_0 \vec{H} \cdot \vec{n} + \cdots \right) \tag{14}$$

where $\vec{n}$ is a local vector of unit length pointing in the direction of the magnetization, $\vec{A}(\vec{n})$ is the field that implements the Berry's phase needed to obtain the correct quantum commutation relations between the spin components, and $\vec{H} = g^* \mu_B \vec{B}$ is the Zeeman field ($|\vec{H}| = E_Z$). The two crucial parameters which enter the action are M_0, the magnetization density, and ρ_s, the spin stiffness. There are other omitted terms in the action, which produce at most a logarithmic correction to the magnetization in two dimensions [14]. There are various ways in which one can proceed at this point [49], but the most convenient one for our purposes involves mapping the spin problem to a problem with Schwinger bosons [50], and making the large-N approximation [50]. There are two common ways of mapping the spin problem to Schwinger bosons: the SU(N) approach and the O(N) approach [14], which give slightly different answers.

In the following, we will restrict the theoretical results to the leading large-N approximation, where the magnetization is given as $M(T) = M_0 \Phi_M(r, h)$, where $r = \rho_s/T$ and $h = E_Z/T$ are scaling variables, and Φ_M is a scaling function. In this approximation, the results of Read and Sachdev [14] for the magnetization in the SU(N) approach are

$$\Phi_M(r, h) = \frac{\log(q_1 - \mathrm{e}^{-h/2}) - \log(q_1 - \mathrm{e}^{h/2})}{8\pi r} \tag{15}$$

and $q_1 > 1$ is a root of the equation

$$(q_1 - \mathrm{e}^{-h/2})(q_1 - \mathrm{e}^{h/2}) = q_1^2 \mathrm{e}^{-8\pi r}. \tag{16}$$

The corresponding results in the leading O(N) approximation [14] are

$$\Phi_M(r, h) = \frac{\log(q_2 - \mathrm{e}^{-h}) - \log(q_2 - \mathrm{e}^{h})}{4\pi r} \tag{17}$$

where $q_2 > 1$ is the solution of

$$(q_2 - \mathrm{e}^{-h})(q_2 - 1)(q_2 - \mathrm{e}^{h}) = q_2^3 \mathrm{e}^{-4\pi r}. \tag{18}$$

The principal assumption underlying the CQFM description is that long-wavelength ferromagnetic spin fluctuations are the only low-energy modes that affect the spin polarization in the temperature range of interest. In the regime where this assumption holds for the $\nu = 1/3$ state, it should be possible to map the CF Hamiltonian theory to the CQFM. Conceptually, one can think of 'integrating out' the fermions and leaving behind an effective theory of the spin fluctuations. Operationally, one needs to find the values of M_0 and ρ_s corresponding to the $\nu = 1/3$ state. An additional complication arises here: since the underlying fermionic theory responds to temperature by self-consistently modifying occupations and energies, one should expect to obtain temperature-dependent values $M_0(T)$ and $\rho_s(T)$. We will extract these values from the CFHF Hamiltonian results.

First consider $M_0(T)$. We already have a value for the CFHF magnetization $M_{HF}(T)$. Let us first write down the correct relation between M_0 and M_{HF} and then justify it:

$$M_{HF}(T) = M_0(T) \Phi_M(r = 0, h = \Delta_{SR}/T). \tag{19}$$

In the CFHF theory the particles and holes are treated as independent, or noninteracting, with a gap equal to Δ_{SR} (to be more precise, this is the lowest-energy spin-flip excitation). To

put it differently, the energy gap required to create this spin-flip excitation is always Δ_{SR}, no matter what the distance is between the particle and the corresponding hole. This is as though the collective mode dispersion were completely flat, $\omega(q) = \Delta_{SR}$. The CQFM description that corresponds most closely to the CFHF one is the one that has *the same spin-flip excitation spectrum*, namely one with no spin stiffness, that is, $r = 0$, and an effective Zeeman field $E_Z^{eff} = \Delta_{SR}$. The CQFM prediction for the magnetization of such a theory is the right-hand side of equation (19), which has to be equated to the CFHF prediction; hence the above formula, equation (19).

Knowing that, for SU(N),

$$\Phi_M(0,h) = \tanh(h/2) \tag{20}$$

while, for O(N),

$$\Phi_M(0,h) = \frac{\sinh(h)}{1/2+\cosh(h)} \tag{21}$$

one extracts $M_0(T)$ from equation (19), using the value of $M_{HF}(T)$ computed in section 3†.

Next consider the spin stiffness $\rho_s(T)$. At a given temperature T the self-consistent occupations $N_{F,GS}(\sigma,n)$ and energies $\epsilon(\sigma,n)$ in the ground state are computed using the procedure described in section 2. Now one creates a twisted spin state by defining

$$\begin{aligned} d_{\uparrow,n,X} &= \cos(qX/2)d_{\alpha,n,X} + \sin(qX/2)d_{\beta,n,X} \\ d_{\uparrow,n,X} &= \cos(qX/2)d_{\alpha,n,X} + \sin(qX/2)d_{\beta,n,X} \end{aligned} \tag{22}$$

where α, β define *local* directions of up and down. In the twisted spin state the occupations of the local up and down spins remain the same as in the ground state, that is

$$\begin{aligned} \langle d^\dagger_{\alpha,n,X} d_{\alpha,n',X'}\rangle &= \delta_{nn'}\delta_{XX'}N_{F,GS}(\uparrow,n) \\ \langle d^\dagger_{\beta,n,X} d_{\beta,n',X'}\rangle &= \delta_{nn'}\delta_{XX'}N_{F,GS}(\downarrow,n) \\ \langle d^\dagger_{\alpha,n,X} d_{\beta,n',X'}\rangle &= 0. \end{aligned} \tag{23}$$

This leads to the following expectation values for the actual (global) $\uparrow$ and $\downarrow$ spin directions:

$$\begin{aligned} \langle d^\dagger_{\uparrow,n,X} d_{\uparrow,n',X'}\rangle &= \delta_{nn'}\delta_{XX'}(\cos^2(qX/2)N_{F,GS}(\uparrow,n) + \sin^2(qX/2)N_{F,GS}(\downarrow,n)) \\ \langle d^\dagger_{\downarrow,n,X} d_{\downarrow,n',X'}\rangle &= \delta_{nn'}\delta_{XX'}(\sin^2(qX/2)N_{F,GS}(\uparrow,n) + \cos^2(qX/2)N_{F,GS}(\downarrow,n)) \\ \langle d^\dagger_{\uparrow,n,X} d_{\downarrow,n',X'}\rangle &= \delta_{nn'}\delta_{XX'}\sin(qX/2)\cos(qX/2)(N_{F,GS}(\uparrow,n) - N_{F,GS}(\downarrow,n)) \\ \langle d^\dagger_{\downarrow,n,X} d_{\uparrow,n',X'}\rangle &= \langle d^\dagger_{\uparrow,n,X} d_{\downarrow,n',X'}\rangle \end{aligned} \tag{24}$$

and corresponds to a state where the unit vector $\vec{n}$ pointing in the direction of the local magnetization has components $\vec{n} = (\sin(qX), 0, \cos(qX))$. Using the expectation values from equation (24), one can compute the HF energy of the twisted ground state, and thence the excess energy to order q^2. Comparing to the energy cost of a twist in the CQFM, which is $(\rho_s/2)L^2q^2$, one finds the spin stiffness

$$\begin{aligned} \rho_s = \frac{1}{16\pi}\int \frac{\mathrm{d}^2 s}{(2\pi)^2}\, v(s) \sum_{n_1,n_2} |\rho_{n_1 n_2}(s)|^2 \\ \times (N_F(\uparrow,n_1) - N_F(\downarrow,n_1))(N_F(\uparrow,n_2) - N_F(\downarrow,n_2)) \end{aligned} \tag{25}$$

† Note that neither of the above formulae agrees with the result for itinerant noninteracting electrons at $\nu = 1$, namely, $\Phi_M = \tanh(h/4)$. However, since we will be using the large-N forms with nonzero r later, we have to use the above forms for consistency.

where L^2 is the area of the system, and $\rho_{n_1 n_2}$ is the density matrix element of equation (6). One caveat should be mentioned here: the above should be regarded as an estimate for the twist rather than a rigorous calculation (even in the HF approximation), since ideally one should compute the *free-energy* cost of a twist, rather than just the *internal energy* cost, as we have done. The free-energy cost would be computed by carrying out a self-consistent HF calculation at finite T in the presence of a twist. However, a fully relaxed HF calculation of an inhomogeneous state at $T \neq 0$ is computationally prohibitive. In the following we use the value given by equation (25).

Figure 8 shows the results for $M_0(T)$ and $\rho_s(T)$ for the parameters corresponding to the Melinte *et al* M242 sample [12]. As can be seen, ρ_s in particular has a dramatic T-dependence, and essentially vanishes for $T > 6$ K. This corroborates the earlier observation that the spin-reversed gap collapses to E_Z at around the same temperature, as seen in figure 4.

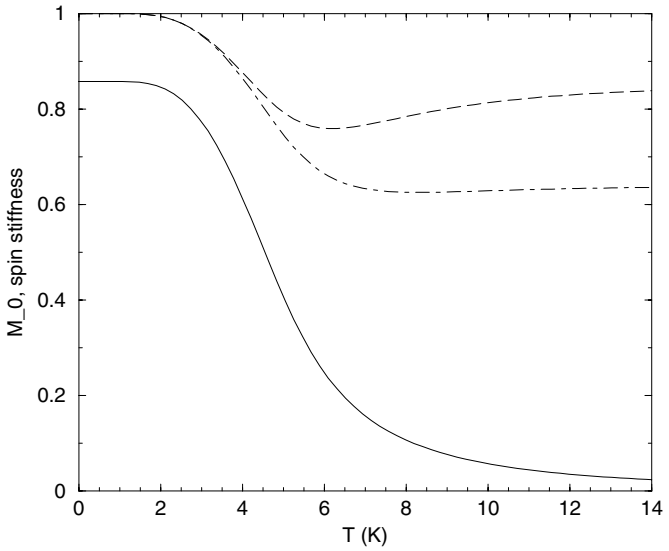

Figure 8. The parameters $M_0(T)$ (normalized to 1 at $T = 0$) for SU(N) (dashed line) and O(N) (dot–dashed line), and $\rho_s(T)$ ($\times 400$ in units of E_C) extracted from the CFHF approximation for $E_Z = 0.0186 E_C$ and $\lambda = 1.5l$. While M_0 shows no dramatic behaviour, ρ_s decreases precipitously beyond 3 K.

Having extracted the parameters $M_0(T)$ and $\rho_s(T)$, it is now easy to calculate the effects of spin fluctuations in the leading large-N approximation from equations (15), (17). Figure 1 (in the introduction) presents the results for the Khandelwal *et al* data [11]. As can be seen, the agreement between the theory and experiment is now quite good, indicating that the proper treatment of long-wavelength thermal spin fluctuations has remedied most of the defects of the CFHF prediction. Turning to the M242 sample of Melinte *et al* [12], a similar dramatic improvement in the agreement between theory and experiment is seen in figure 2 (in the introduction). There are still some discrepancies in both the figures, but they never amount to more than 10–15%. The agreement with exact diagonalizations [48] is now also excellent up to about 6 K (comparing our results for the Khandelwal *et al* [11] 10W sample to the $g = 0.5$ line in figure 1 of reference [48]). Beyond 6 K, our results drop below the exact-diagonalization results and are a better fit to the experiments.

Note that we have set the thickness at $\lambda = 1.5l$. How do changes in λ affect the results? As seen from the figures in the earlier short exposition of the CFHF approximation [30], the

changes in the CFHF results are not dramatic: small changes in λ lead to small changes in the results. Generically, as λ increases, the gaps, the spin stiffness, and the CFHF polarization (at any given T) all decrease smoothly [30]. However, once the fluctuation effects are folded in, the results seem *very insensitive* to λ . For example, the results including spin fluctuations are indistinguishable by eye for $\lambda = 1.5$ and $\lambda = 1.75$. Thus, while a value of $\lambda = 1.5$ has been used for illustrative purposes, the results are very robust with respect to small changes in λ.

One might wonder how important it is to keep the temperature dependence of the parameters M_0 and ρ_s. Figure 9 shows the prediction of the large-N approach using the values of M_0 and ρ_s computed at $T = 0$ for the parameters corresponding to the Khandelwal *et al* data [11]. This fit ignores finite-temperature single-particle excitations. It is clear that while this prediction is somewhat better than the CFHF result for lower temperatures, it is *worse* beyond 6 K. Also, this prediction is uniformly worse than the one including both CFHF and spin-fluctuation effects.

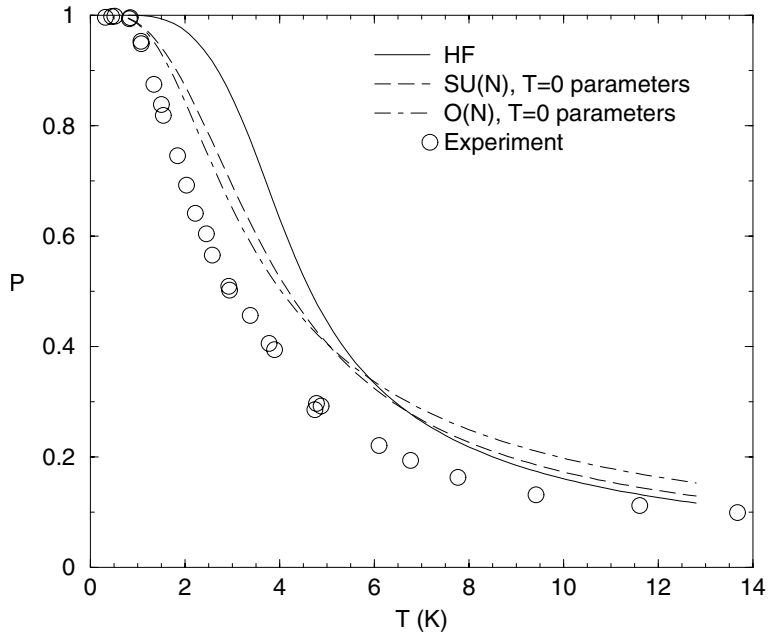

Figure 9. The theoretical prediction for the Khandelwal *et al* data [11] based on a zero-temperature fit to the parameters of the CQFM. The thickness parameter is assumed to be $\lambda = 1.5l$. Circles represent the experimental data [11].

Clearly, the reason is that this fit does not incorporate the finite-temperature fermionic single-particle effects that renormalize the spin stiffness substantially downwards beyond about 3 K, as seen in figure 8. It must be concluded that the temperature dependence of the parameters M_0 and ρ_s is crucial, and that *both single-particle and thermal spin-wave effects must be treated correctly* if one is to accurately predict the spin polarization.

5. Conclusions, caveats, and open questions

In summary, an approximate analytical method for computing the temperature dependence of the polarization for an arbitrary fractional quantum Hall state has been presented, and illustrated for $\nu = 1/3$ and $2/5$. It consists of two steps. The first step is constructing a

finite-temperature composite-fermion Hartree–Fock approximation of the CF Hamiltonian. This already incorporates important effects on the polarization resulting from particle–hole excitations, and is sufficient to produce all the correct qualitative features, such as the nonmonotonicity of the polarization for 2/5 [48]. For the spontaneously polarized 1/3 state, however, there is a substantial discrepancy between this prediction and experiment at intermediate temperatures due to the incorrect treatment of spin fluctuations. The second step, for the $\nu = 1/3$ state, is to map the long-wavelength low-energy physics onto the continuum quantum ferromagnet [14]. The parameters that enter the effective theory are extracted from the CFHF approximation. Finally, the approximate solution for a continuum quantum ferromagnet in the large-N approach [14] is used to produce a prediction for the spin polarization which incorporates the effects of both single-particle and spin-wave excitations. Note that the reduction of the spin polarization due to single-particle and spin-wave effects is not additive in the two effects: the parameters M_0 and ρ_s extracted from the CFHF approximation encode information about single-particle excitations in a very nonlinear way. The resulting prediction is in quite good agreement with experimental data, as seen in figures 1 and 2. The comparison between theory and experiment was made using the model potential of equation (3) for illustrative purposes, with a thickness parameter $\lambda = 1.5l$, which ought to be in the physical regime for most samples [44]. However, the above two-step procedure of first carrying out the CFHF approximation, and then mapping the problem onto the CQFM, can be executed for *any* potential $v(q)$.

A number of caveats must be noted at this point. In order for the mapping to the CQFM to make sense, spin-wave excitations must be the only low-energy spin-flip excitations. However, as T increases, other spin-flip excitations also become relevant (such as a collective excitation from the $\uparrow$-spin $n = 0$ CF-LL to the $\downarrow$-spin $n = 1$ CF-LL). Another complicating issue is that while the CQFM model assumes the spin-wave dispersion to be quadratic at all q, in reality it turns over and becomes asymptotic at the spin-reversed gap Δ_{SR}. Both of these considerations suggest that the mapping is to be trusted only in the regime where $T \leqslant \Delta_{SR}(T)$. One can verify from figure 5 that this corresponds to $T \leqslant 6$ K or so, which fortunately includes the most interesting region of the data. It is somewhat puzzling that the prediction from the mapping works well to twice this temperature. Finally, $1/N$ corrections can be expected to the leading large-N prediction, which reduce the magnetization slightly beyond the leading large-N result (see, for example, Timm *et al* [14]). These corrections would improve the agreement between the theory and the data at low T.

An important open problem is the development of a formalism in which one can systematically integrate out fermions, leaving behind a theory of the low-energy excitations, perhaps along the lines of reference [15]. Such excitations need not be restricted to spin-wave excitations, but may perhaps include other spin-flip modes or even density modes, depending on the temperature and the system. Such a theory is necessary to address the effects of thermal spin fluctuations on the temperature-dependent spin polarization of fractional Hall states which are *not* fully polarized in the absence of a Zeeman coupling. The spin-flip modes of such systems do not have the simple quadratic CQFM form even at small q (for example, the spin-flip collective mode of the fully polarized 2/5 state starts with a *negative* quadratic term [29]). A further interesting application would be to the interplay of spin and density fluctuations in the compressible fractional Hall states [26, 43].

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